University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO EXAMINATION 2

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (60 points)
A block of mass m slides without friction on a wedge. The top surface of the wedge is at an angle α with respect to the horizontal. Denote by s the distance along the surface of the wedge between the mass and the tip of the wedge. Under the influence of gravity, $\ddot{s}$ will be negative (its value is not given).

(**a**) (5 points)
Write the potential energy $U(s)$ of the block as a function of s, defining $U(0) \equiv 0$.
Solution:

$$U = mgs\sin\alpha \ .$$

(**b**) (5 points)
For this part, assume that the wedge is fixed. Write the kinetic energy T of the block as a function of $\dot{s}$.
Solution:

$$T = \tfrac{1}{2}m\dot{s}^2 \ .$$

(**c**) (10 points)
For the conditions of part (**b**), use s as the generalized coordinate and solve the Euler-Lagrange equation for $\ddot{s}$.
Solution:

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}m\dot{s}^2 - mgs\sin\alpha \\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} &= \frac{\partial\mathcal{L}}{\partial s} \\ m\ddot{s} &= -mg\sin\alpha \\ \ddot{s} &= -g\sin\alpha \ . \end{aligned}$$

(**d**) (10 points)
For the remainder of this problem, assume that the wedge is allowed to slide without friction on a horizontal table. Take the wedge's horizontal coordinate to be x and its mass to be M. The block's coordinate s continues to be measured relative to the *wedge*, not the table. The kinetic energy of the block-wedge system becomes

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\dot{s}^2 + \dot{x}^2 - 2\dot{x}\dot{s}\cos\alpha\big) \ .$$

Write the Euler-Lagrange equation(s) using s and x as generalized coordinates.
Solution:

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\dot{s}^2 + \dot{x}^2 - 2\dot{x}\dot{s}\cos\alpha\big) \\ & - mgs\sin\alpha \ . \end{aligned}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial s} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} \\ -mg\sin\alpha &= \frac{d}{dt}\big(m\dot{s} - m\dot{x}\cos\alpha\big) \\ -mg\sin\alpha &= m\ddot{s} - m\ddot{x}\cos\alpha \ . \end{aligned}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ 0 &= \frac{d}{dt}\big((M+m)\dot{x} - m\dot{s}\cos\alpha\big) \\ 0 &= (M+m)\ddot{x} - m\ddot{s}\cos\alpha \ . \end{aligned}$$

(**e**) (15 points)
Find the cyclic coordinate and the conserved generalized momentum p that is canonically conjugate to it. If the block and the wedge are released from rest, what is the value of p?
Solution:
The cyclic coordinate is x, because $\frac{\partial \mathcal{L}}{\partial x} = 0$. [Even if you do not remember what a cyclic coordinate is, you can see from the answer to (**d**) that

$$p = (M+m)\dot{x} - m\dot{s}\cos\alpha$$

is conserved; the generalized coordinate that is conjugate to this canonical momentum is x.] The system is released from rest, so $\dot{s}$ and $\dot{x}$ are zero at that time. Therefore, being conserved, $p = 0$ for all subsequent times.
(**f**) (15 points)
Using the non-cyclic Euler-Lagrange equation and the result of (**e**), solve for $\ddot{s}$. [*Hint:* If $\alpha = 30°$ and $M = m/2$, your answer should be twice what you found for (**c**).]
Solution: From the E-L equation in x,

$$\ddot{x} = \frac{m}{M+m}\ddot{s}\cos\alpha\ .$$

Substituting this into the E-L equation in s,

$$\ddot{s} - \frac{m}{M+m}\ddot{s}\cos^2\alpha = -g\sin\alpha$$
$$\ddot{s} = \frac{-g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha}\ .$$

2. (40 points)
A mass m is connected to a fixed wall by a massless spring of constant k. The coordinate of the mass is $x = 0$ when the spring is relaxed. Neglect gravity.
(**a**) (10 points)
Assume that there is no damping force on m. At $t = 0$, m satisfies the initial conditions $x(0) = -x_0$, where x_0 is a positive constant, and $\dot{x}(0) = 0$. Write down $x(t)$ for $t > 0$.
Solution:

$$x(t) = -x_0\cos\omega_0 t$$
$$\omega_0 \equiv \sqrt{\frac{k}{m}}\ .$$

(**b**) (10 points)
For the conditions of (**a**), denote the force of constraint exerted on the spring by the wall to be $F(t)$ (a function of time that is unknown until you figure it out). What is the earliest positive time $t_0 > 0$ at which $F(t_0)$ vanishes?
Solution:
The spring is massless, so there is no net force on it (otherwise it would have infinite acceleration). Therefore the force of the wall on the spring is equal to the force of the spring on the mass. The latter is just $m\ddot{x}$, which first vanishes at

$$\cos\omega_0 t_0 = \pi/2$$
$$t_0 = \frac{\pi}{2\omega_0}\ .$$

(**c**) (10 points)
Now assume that there exists an additional force $-b\dot{x}$ on m due to damping. For the initial conditions of (**a**), find the value approached by $x(t)$ in the limit $t \to \infty$.
Solution:
All homogeneous solutions (underdamped, overdamped, critically damped) contain a factor $\exp(-Ct)$ where C is some positive constant. Each possible solution will therefore vanish at $t \to \infty$, so $x(\infty) \to 0$.
(**d**) (10 points)
For the conditions of (**c**), it is observed that x never oscillates if the damping constant b is larger than a minimum value b_{min}. What is b_{min}? Explain the reasoning behind your answer.
Solution: The only homogeneous solution that oscillates (in addition to being damped) is the underdamped solution, which requires $Q > \frac{1}{2}$. Therefore, since there is no oscillation,

$$Q \le \tfrac{1}{2}$$
$$\frac{\omega_0}{\gamma} \le \tfrac{1}{2}$$
$$\frac{\omega_0}{b/m} \le \tfrac{1}{2}$$
$$2\omega_0 m \le b$$
$$2\sqrt{km} \le b\ .$$